Dilip Kumar Sobyalthi

Generalised Co-ordinates:

In order to describe the configuration of a system, some variables are necessary, usually called Co-ordinates. These are not necessarily our usual Cartisian Co-ordinate but may be any other variables such as angles, radial vectors etc. The requirement is that, with these variables we must conviniently describe the system. Any such set of variables are called generalised co-ordinates. Such Coordinates gives us reasonable mathematical simplification of a given problem in physics.

Notation: Usually generalised co-ordinates are represented by letter q's. ie. $q_1, q_2 \cdots q_n$. or $q_j\ (j = 1, 2, \cdots n)$. When we work out in a known system the symbols $q_1, q_2 \cdots$ take the place of corresponding co-ordinates.

Ex. i) For cartisian Co-ordinates $q_1 = x, q_2 = y, q_3 = z$

ii) For plane polar co-ordinates $\begin{cases} q_1 = r, \ q_2 = \theta \\ \text{or } q_1 = x, \ q_2 = y \end{cases}$

Transformation Equation Generalised $\leftrightarrow$ Cartisian Co-ordinates

It is trivial to represent the generalised Co-ordinates in terms of cartisian Co-ordinates, since we are accustomed to the later system.

$$q_1 = q_1(x_1, y_1, z_1, x_2, y_2, z_2 \cdots x_N, y_N, z_N, t)$$

$$\vdots$$

$$q_{3N} = q_{3N}(x_1, y_1, z_1 \cdots \cdots x_N, y_N, z_N, t^*)$$

(* Every coordinate is explicit f^n of time). — (1)

The above eqn (1) are the set of transformation for a N particle system (Constraint-free) which need $3N$ generalised co-ordinates.

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The inverse transformation eqn are

$$x_1 = x_1(q_1 q_2 \cdots q_{3N}, t)$$
$$y_1 = y_1(\quad \cdots \quad)$$
$$z_1 = z_1(\quad \cdots \quad)$$
$$\vdots$$
$$z_N = z_N(q_1, q_2 \cdots q_{3N}, t) \quad \text{——— (2)}$$

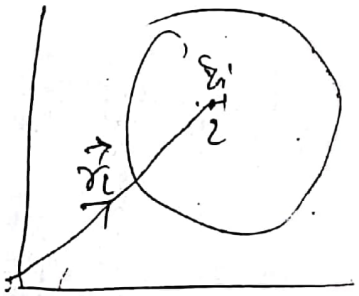
However the only requirement that the above set [either (1) or (2)] represent a transformation $q_j\,(j = 1, 2, 3 \cdots 3N) \longleftrightarrow (x_1, y_1, z_1, \cdots, z_N)$ is that the Jacobian determinant

$$J\left\{\frac{q_1, q_2 \cdots q_{3N}}{x_1, y_1, \cdots z_N}\right\} = \left\{\frac{\partial(q_1 \cdots q_{3N})}{\partial(x_1, \cdots z_N)}\right\}$$

$$= \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_{3N}}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \cdots & \cdots & \frac{\partial q_{3N}}{\partial x_2} \\ \vdots & & & \\ \frac{\partial q_1}{\partial z_N} & \frac{\partial q_2}{\partial z_N} & \cdots & \frac{\partial q_{3N}}{\partial z_N} \end{vmatrix} \neq 0.$$

Generalized Notation:

Generalized displacement:



Consider a N particle system.

$\vec{r}_i \rightarrow$ pos^n vector of i^{th} particles.

$\delta\vec{r}_i \rightarrow$ be a small displacement of $\vec{r}_i\,(i = 1, 2 \cdots N)$ (called virtual displacement) with time 't' held fixed

We know that $\vec{r}_i$ are f^n of $q_1 q_2 \cdots q_{3N}, t$ i.e.

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \cdots q_{3N}, t) \quad \text{——— (3)}$$

Small displacement

$$\delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_i}{\partial q_2}\delta q_2 \cdots + \frac{\partial r_i}{\partial q_{3N}}\delta q_{3N} + \frac{\partial\vec{r}_i}{\partial t}\delta t$$

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$$\therefore \delta\vec{r}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \quad \text{---- (4)}$$

In eq'n (4) the δq_j are called generalised displacement.

Generalised velocity

The time derivative of q_j's are called generalised velocity.

We have from (3)

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_{3N}, t)$$

$$\Rightarrow \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \vec{r}_i}{\partial q_{3N}}\frac{dq_{3N}}{dt} + \frac{\partial \vec{r}_i}{\partial t}\frac{dt}{dt}$$

$$= \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial \vec{r}_i}{\partial t}$$

$$= \sum_{j=}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \quad \text{---- (5)}$$

$\dot{q}_j$ are called the generalised velocity.

Generalised Force:

Consider work done by force $\vec{F}_i$ during a small virtual displacement $\delta\vec{r}_i$ is

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta r_i = \sum_i \vec{F}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

$$= \sum_j \left[\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right]\delta q_j$$

$$= \sum_j Q_j \delta q_j \quad \text{---- (6)}$$

Here we may identify $\vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j} = Q_j$ as the generalised force ---- (7)

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Generalised potential

(a) For conservative system —

The forces are derivable from scalar pot. V (which is a f^{n} of pos^{n} alone). V represents the P.E. of the system. So work done

$$\delta W = -\delta V \quad \text{(Change in P.E)}$$

$$= -\sum_{j=1}^{3N} \frac{\partial V}{\partial q_j} \delta q_j$$

$$= \sum_j Q_j \delta q_j \quad \text{with } Q_j = \frac{-\partial V}{\partial q_j} \text{ being the generalised force} \quad (2).$$

(b) For non conservative system:

$$Q_j = \frac{-\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$$

V — are called vel. dependent potential or simply — generalised potential for non conservative system.

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Deduction of Hamilton's Principle from D'Alembert Principle:

According to D'Alembert Principle

$$\boxed{\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0}$$

$$\Rightarrow \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \vec{F}_i \cdot \delta\vec{r}_i \quad \text{——— (*)}$$

$$\Rightarrow \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta r_i$$

Now $$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \frac{d\vec{p}_i}{dt} \cdot \delta r_i = \sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \delta\vec{r}_i$$

$$= \sum_i \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt}\right) \cdot \delta r_i$$

$$= \sum_i \frac{d}{dt}\left[m_i \frac{d\vec{r}_i}{dt} \cdot \delta r_i\right] - m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d}{dt}(\delta\vec{r}_i)$$

Since δ & d are operators and can commute ↘

$$\delta\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\delta\vec{r}) \qquad \left| \delta(dx) = d(\delta x) \right.$$

So

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt} \cdot \delta\vec{r}_i\right) - m_i \frac{d\vec{r}_i}{dt} \delta\left(\frac{d\vec{r}_i}{dt}\right)$$

$$= \sum_i \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt} \cdot \delta\vec{r}_i\right) - \delta\left(\frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt}\right)^2\right) \quad *$$

$$= \sum_i \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt} \cdot \delta r_i\right) - \delta\left(\frac{1}{2} m_i v_i^2\right)$$

Using m (*)

$$\sum_i \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt} \cdot \delta r_i\right) - \delta(T) = \sum_i f_i \cdot \delta r_i \qquad \left| T = \frac{1}{2} m_i v_i^2 = K.E \right.$$

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$$\Rightarrow \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt}\cdot \delta r_i\right) = \sum_i \vec{F}_i \cdot \delta\vec{r}_i + \delta(T)$$

For a Conservative System $\sum F_i \cdot \delta r_i = -\delta V$

(work done $\delta W = F_i \cdot \delta r_i = -\delta V$) Change in

So $\frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt}\cdot \delta\vec{r}_i\right) = \delta T - \delta V = \delta(T-V)$

Integrating wrt time from $t_1 \to t_2$

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(m_i \frac{d\vec{r}_i}{dt}\cdot \delta\vec{r}_i\right) dt = \int_{t_1}^{t_2} \delta(T-V)\, dt$$

$$\Rightarrow \left[m_i \frac{d\vec{r}_i}{dt}\cdot \delta\vec{r}_i \right]_{t_1}^{t_2} = \delta \int_{t_1}^{t_2} (T-V)\, dt$$

In δ variation method $\delta r_i \Big|_{t_1}^{t_2} = 0$ since there is no variation at the end points of any path because all paths are traversed in same time.

So $m_i \frac{d\vec{r}_i}{dt}\cdot \delta\vec{r}_i \Big|_{t_1}^{t_2} = 0$

$$\Rightarrow \delta \int_{t_1}^{t_2} (T-V)\, dt = 0$$

or $\delta \int_{t_1}^{t_2} L\, dt = 0$. This is

Hamilton's Principle.

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Hamiltonian formulation

Lagrange Eq" of motion are a set of 2nd order differential eqn involving generalised co-ordinates. Hamiltonian formulation is a new alternative approach for solving dynamical problems. It is a basic foundation of statistical and quantum mechanics.

Generalised Momentum and Cyclic Co-ordinate:

Consider a particle in linear motion along x axis. The K.E of the particle is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \quad \text{——(1)}$$

$$\Rightarrow \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p \quad \text{——(2)}$$ defines the momentum of the particle.

If the system is conservative i.e. ~~velocity is~~ potential energy is velocity independent but, Then

$$V = V(x) \Rightarrow \frac{\partial V}{\partial \dot{x}} = 0 \quad \text{——(3)}$$

Combining (2) and (3)

$$\frac{\partial T}{\partial \dot{x}} - \frac{\partial V}{\partial \dot{x}} = \frac{\partial (T-V)}{\partial \dot{x}} = p$$

$$\text{or } \frac{\partial L}{\partial \dot{x}} = p \quad \text{——(4)}$$ defines the momentum in a new way.

For a system described by q_k (gen. co-ordinate) & gen. velq. we define the generalised momentum by

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \text{——(5)}$$

Called conjugate/canonical momentum.

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~~This can also be obtained from the~~

Now Lagrange's Eqn (for a conservative sys) conserv

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

put $\frac{\partial L}{\partial \dot{q}_k} = p_k$, then $\frac{d}{dt}(p_k) - \frac{\partial L}{\partial q_k} = 0$

or $\dot{p}_k = \frac{\partial L}{\partial q_k}$

Gen.

If the Lagrangian $L(q_k, \dot{q}_k, t)$ does not ~~contain~~ have explicit dependence on q_k i.e. it appears as $L(\dot{q}_k, t)$, then

$$\frac{\partial L}{\partial q_k} = 0$$

$$\Rightarrow \dot{p}_k = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

$$\text{or} \quad p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{is a constant} \qquad (6)$$

So when the Lagrangian does not contain a co-ordinate q_k, the generalised momentum p_k is a constant of motion.

The co-ordinate q_k are called Cyclic/ignorable co-ordinate.

Ex: For a particle in central force field (in polar coord

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$x = r\cos\theta$
$y = r\sin\theta$
$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$
$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$
$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$
$= \frac{1}{2}(m\dot{r}^2 + r^2$

The lagrangian does not contain θ. So θ is a Cyclic co-ordinate. The generalised momenth

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{is const and is the angular momentum, which is const. with time.}$$

So Ang. mom is a const. in central force motion is called 1st integra

and $\frac{\partial L}{\partial \dot{\theta}} = 0 = p_\theta = \text{Const}$ as obtained by integrating $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$

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Conservation of Energy / Hamiltonian function / Jacobi Integral:

For a conservative system, the general Lagrangian is given by

$$L = L(q_k, \dot{q}_k, t) \quad k = 1, 2 \cdots n. \qquad \text{---(7)}$$

Taking the time derivative

$$\frac{dL}{dt} = \sum_k \left(\frac{\partial L}{\partial q_k} \cdot \frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k} \frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \right)$$

$\left(= \dot{q}_k\right)$

The Lagrange eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$ yields $\qquad \text{---(8)}$

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right); \text{ substituting in (8)}$$

$$\frac{dL}{dt} = \sum_k \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \right]$$

$$= \sum_k \left\{ \frac{d}{dt}\left[\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right] + \frac{\partial L}{\partial t} \right\}$$

$$\Rightarrow \quad \sum_k \left\{ \frac{d}{dt}\left(\underline{\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L} \right) = -\frac{\partial L}{\partial t} \right. \qquad \text{---(9)}$$

The underlined term within the simple bracket has the dimension of energy and is called energy function, denoted in general by 'h' i.e.

$$\sum_k \left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = h(q_k, \dot{q}_k, t) \qquad \text{---(10)}$$

With this notation (9) becomes

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \qquad \text{---(11)}$$

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If the Lagrangian has no explicit time dependance, then $\frac{\partial L}{\partial t} = 0$. So we have

$$\frac{dh}{dt} = 0 \Rightarrow h = \text{is a const} \quad \text{---(12)}$$

Thus when the Lagrangian is not an explicit function of time, the energy function 'h' is the constant of motion.

This is one of the first integrals of motion (as it is obtained by integrating $\frac{dh}{dt}$ wrt. time) and is called Jacobi Integral.

Now the energy function can be written as (from (11))

$$\frac{dh(q_k, \dot{q}_k, t)}{dt} = \frac{-\partial L}{\partial t}$$

or $$\frac{d}{dt}\left(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = \frac{-\partial L}{\partial t}$$

with $\frac{\partial L}{\partial \dot{q}_k} = p_k$ being the generalised momentum, we have

$$\frac{d}{dt}\left(\sum_k p_k \dot{q}_k - L\right) = \frac{-\partial L}{\partial t} \quad \text{---(13)}$$

The qty or $\boxed{\sum_k p_k \dot{q}_k - L = H}$ is called the Hamiltonian function, which is general is a function of generalised momenta 'p_k', generalised co-ordinate(s) 'q_k' and time 't'. ---(14)

So $$H = H(p_k, q_k, t). \quad \text{---(14)}$$

(The energy fn 'h' and Hamiltonian fn H are identical except that H is a fn of p_k, q_k, t whereas 'h' is a fn of $q_k, \dot{q}_k, t$).

If the Lagrangian does not depend on time explicitly then $\frac{\partial L}{\partial t} = 0$ So from (13) $\frac{dH}{dt} = 0$ or

$H = \sum_k p_k \dot{q}_k - L$ is a constant of motion.

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Thus if 't' doesn't appear in L, the Hamiltonian 'H' is const in time / conserved. In most cases the Hamiltonian is equal to the total energy (E) of the system.

Conservation of Energy:

For a conservative system the P.E V is ind. of velocity $\dot{q}_k$. The transformation eq^n for co-ordinates is

$$\vec{r}_i = \vec{r}_i(q_k) \quad \text{(do not contain time explicitly)}.$$

for such system $\frac{\partial V}{\partial \dot{q}_k} = 0$

We have $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$

with this, the Hamiltonian H defined by (14) becomes

$$H = \sum_k p_k \dot{q}_k - L = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k - L \qquad \text{——(15)}$$

Since $\vec{r}_i$ does not depend on time explicitly, the K.E. T is a homogeneous quadratic function of $\dot{q}_k$

The generalized vel. $\dot{q}_k$ are given by ——

$$\vec{v}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \to 0 \quad \text{(for conservative syst)}$$

Now $T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right)^2$

$$= \sum_i \frac{1}{2} m_i \left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right) \left(\sum_l \frac{\partial \vec{r}_i}{\partial q_l} \dot{q}_l \right)$$

$$= \sum_i \frac{1}{2} m_i \sum_k \sum_l \left(\frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial q_l} \right) \dot{q}_k \dot{q}_l$$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_l} = \alpha_{kl} = \text{(new fn.)}$$

$T = \sum_k \sum_l \alpha_{kl} \dot{q}_k \dot{q}_l$ is a homogeneous quadratic fn of $\dot{q}_k$'s.

This part of diagram is optional.

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Hamiltonian

According to Euler theorem, if f is a homogeneous f^n of order n, of a set of variable q_k, then

$$\sum_k q_k \frac{\partial f}{\partial q_k} = nf$$

Sin T is a homogeneous f^n of $\dot{q}_k$ of order 2 ($\dot{q}_k$

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T$$

$$\text{or } \sum_k \dot{q}_k p_k = 2T.$$

Using this in 15

$$H = 2T - L = 2T - T + V = T + V = E$$

The total energy and is conserved. (16)

Solhe Hamiltonian represents the total energy E for a conservative system and is constant.

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Hamilton's Eqn. of motion:

The Hamiltonian H in general is a fn of

$$H = H(q_k, \dot{q}_k, t).$$

Since the conjugate momenta is defined by $p_k = \frac{\partial L}{\partial \dot{q}_k}$, we may regard H as a function of gen. co-ordinate q_k, the momenta p_k (or $\dot{q}_k$) and time t so

$$H = H(q_k, p_k, t) \quad \text{——(17)}$$

$$\Rightarrow dH = \sum_k \left[\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \right] \quad \text{——(18)}$$

we define the Hamiltonian H by

$$H = \sum_k p_k \dot{q}_k - L$$

$$\Rightarrow dH = \sum_k \left[\dot{q}_k dp_k + p_k d\dot{q}_k \right] - dL \quad \text{——(19)}$$

Further $L = L(q_k, \dot{q}_k, t)$

$$\Rightarrow dL = \sum_k \left[\frac{\partial L}{\partial q_k} dq_k + \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \right] + \frac{\partial L}{\partial t} dt$$

Using $\frac{\partial L}{\partial q_k} = \dot{p}_k$ and $\frac{\partial L}{\partial \dot{q}_k} = p_k$ we get

$$dL = \sum_k \left(\dot{p}_k dq_k + p_k d\dot{q}_k \right) + \frac{\partial L}{\partial t} dt \quad \text{——(20)}$$

Using (20) in (19) for dL we get

$$dH = \left[\sum_k \left(\dot{q}_k dp_k + p_k d\dot{q}_k \right) \right] - \left[\sum_k \left(\dot{p}_k dq_k + p_k d\dot{q}_k \right) + \frac{\partial L}{\partial t} dt \right]$$

$$dH = \sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \quad \text{——(21)}$$

Comparing (18) and (21) we see

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$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{---- 22(a)}$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} \quad \text{---- 22(b)}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{---- 23}$$

Ex 2

Eqn 22(a) and 22(b) are called Hamilton's Canonical equation of motion

for $k = 1, 2 \ldots n$, the above is a set of 2n 1st order differential equation & much easier to solve (compared to n 2nd order differential eqn of Lagrangian formulation)..

Ex I Harmonic Oscillator

The K.E and P.E are given by

$$T = \frac{1}{2} m\dot{x}^2, \quad V = \frac{1}{2} kx^2$$

Lagrangian $L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

∴ the generalized momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right) = m\dot{x}$$

$$\Rightarrow \dot{x} = p_x/m.$$

$$\therefore \text{K.E } T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m \cdot \frac{p_x^2}{m^2} = \frac{1}{2}\frac{p_x^2}{m}$$

The Hamiltonian $H = T + V = \frac{p_x^2}{2m} + \frac{1}{2} kx^2$.

The Eqn of motion are $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \Rightarrow p_x = m\dot{x}$ ---- (a)

and $-\dot{p}_x = \frac{\partial H}{\partial x} = kx$ ---- (b)

Substituting (a) in (b) LHS of (b)

$$-\dot{p}_x = -\frac{d}{dt}(p_x) = -\frac{d}{dt}(m\dot{x}) = -m\ddot{x} = kx$$

$\Rightarrow m\ddot{x} + kx = 0$ is the eqn of motion of Harmonic Oscillator.

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Ex2 Motion of a particle in a Central force field:

All. Central forces are conservative in nature i.e.

$$F(r) = -\frac{\partial V}{\partial r}$$

(For ex: in case of inverse sq. law of force, $F = -k/r^2 = -\frac{\partial V}{\partial r}$, $V = -$

The K.E $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, P.E $V = V(r)$

Lagrangian $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ —(i)

The conjugate momenta $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = p_r/m$ —(ii)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = p_\theta/mr^2 \text{ (iii)}$$

The Hamiltonian

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= \frac{1}{2}m\left[(p_r/m)^2 + r^2\cdot(p_\theta/mr^2)^2\right] + V_r$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \quad \text{—(iv)}$$

The Hamiltom eq^n of motion, namely $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $-\dot{p}_k = \frac{\partial H}{\partial q_k}$ with ($k = r$ & θ) becomes

$$\dot{r} = \frac{\partial H}{\partial p_r} = p_r/m \text{ —(a)} \quad \text{and} \quad -\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \text{ —(a)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ —(b)} \text{ —(v)} \qquad -\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \text{ —(b)} \text{ —(vi)}$$

from (vi)(b), $-\dot{p}_\theta = 0 \Rightarrow p_\theta = mr^2\dot{\theta}$ = angular momnt is constant in central force motion.

from (ii) $p_r = m\dot{r}$ or $\dot{p}_r = m\ddot{r}$ —(vii)

using (vii) in vi(a) $-m\ddot{r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r}$

or $m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = 0$

or $m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\partial}{\partial r}(-k/r) = 0$

$$\boxed{m\ddot{r} - p_\theta^2/mr^3 + k/r^2 = 0}$$

is the diff^l eq^n of motion in central force field.

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Ex 3 Motion of a Charged particle in an external electric and magnetic field / electromagnetic

The Lagrangian is given by

$$L = T - V = T - (q\dot{Q} - q\vec{V}\cdot\vec{A})$$

q - the mass
Q - scalar
$\vec{V}$ - vel.
$\vec{A}$ - vector

$$\text{or } L = \sum_k \frac{1}{2} m v_k^2 - q\left(\dot{Q} - \sum_k v_k A_k\right) \quad \text{---(i)}$$

The canonical momenta

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial v_k} = m v_k + q A_k \quad \text{---(ii)}$$

The Hamiltonian $H = \sum_k p_k \dot{q}_k - L$

$$= (m v_k + q A_k) v_k - L$$

$$= \sum_k m v_k^2 + q v_k A_k - \frac{1}{2} m v_k^2 + q\left(\dot{Q} - \sum_k v_k A_k\right)$$

$$= \sum_k \frac{1}{2} m v_k^2 + q\dot{Q} \quad \text{---(iii) a.}$$

$$= T + q\dot{Q} \quad \text{---(iii) b}$$

from (ii) $v_k = \frac{p_k}{m} - \frac{q A_k}{m}$

Using this in (iii) a,

$$H = \sum_k \frac{1}{2m}(p_k - q A_k)^2 + q\dot{Q} \quad \text{---(iv)}$$

The Hamiltons Eqn of motion are

$$\dot{v}_k = \frac{\partial H}{\partial p_k} = \frac{1}{m}(p_k - q A_k) \text{ or } \dot{\vec{V}} = \frac{1}{m}(\vec{p} - q\vec{A})$$

$$\dot{p}_k = \frac{-\partial H}{\partial q_k} = -q\nabla\dot{Q} + q\nabla(\vec{V}\cdot\vec{A})$$

— o —

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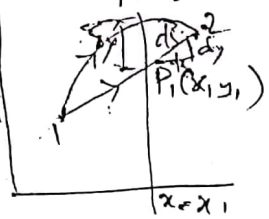
Variational Principle and Lagrangian formulation:

Alternative procedural methods have been developed for solving mechanical problems instead of Newtonian Calculation one such elegant method is the Calculus of variation.

Calculus of variation and its techniques:

In dynamical problems we need to know the position of a particle at any instant of time as well of the path. Such information is obtained by differential methods. But when we need to know about the whole path we apply integral methods and in particular when we need to know about the extremeties we apply calculus of variation.



Consider motion of a particle from posⁿ 1 → 2. The shortest path being the straight line path we represent it by

$$y = mx + c \text{ or } y = y(x)$$

when we are interested with the length of the path, we take a small element ds and integrate it to get the total length I (say) s.t

$$I = \int ds = \int (dx^2 + dy^2)^{1/2} = \int [(dx)^2 (1 + (\tfrac{dy}{dx})^2)]^{1/2}$$

$$= \int (1+\dot{y}^2)^{1/2} dx = \int f(\dot{y}) dx = \int f(y, \dot{y}, x)\, dx \quad \text{(I)}$$

(1)

(∵ y, x are included in the f^n as $\dot{y}$ involve both y & x)

In the above notation I represent the path length and is integral of the f^n "f". If this path I be extremum, then there should be no variation to it i.e. its variation must vanish.

$$\delta I = \delta \int f(y, \dot{y}, x)\, dx = 0. \quad \text{(II)}$$

or $$\delta I = \delta \int f(y(x), \dot{y}(x), x)\, dx = 0$$

However when we are to consider both the paths the eqⁿ $y = y(x)$ is not enough to represent both paths e.g. the eqⁿ $y = mx + c$ represent the straight line path only. So to include other paths some other parameter say 'α' is necessary for designating these paths. So we put $y = y(x, \alpha)$ so that $y = y(x, \alpha_1)$ & $y = y(x$

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represent the two paths. Here α distinguishes path 1, The above relation can be conveniently put as

$$y(x,\alpha) = y(x,0) + \alpha\,\eta(x) \quad ; \quad \text{—(3)}$$

(where $\eta(x)$ is an arbitrary fn of x and vanish at end pts. $\alpha = 0$ in $y(x,0)$ may be taken to represent the extremum path

As in eqn (1) I represent the path length and is diff. for diff paths, it now becomes a fn of α and is given as

$$\boxed{I(\alpha) = \int f[y(x,\alpha), \dot{y}(x,\alpha), x]\,dx} \quad \text{—(4)}$$

* it may be noted that x is independent of α following

$$\frac{\partial x}{\partial \alpha} = 0 \quad \text{and} \quad \text{—(a)}$$

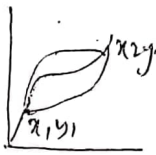
** and at the end points there is no variation in y even i.e.

(2) $$\left.\frac{\partial y}{\partial \alpha}\right|_{\text{at end pt}} = 0 \quad \text{—(b)}$$

*** further the system travels from pt 1 to 2 in same time along any path then

$$\frac{\partial t}{\partial \alpha} = 0 \quad \text{—(c)}$$ along the path and also at the pts.

Now α is that parameter which refer to diff. paths.



so $\frac{\partial I}{\partial \alpha} = 0$ will represent that path for which —(5) the integral I given by (4) is extremum

$$\text{or} \quad \frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right] dx .$$

the limits of integration is fixed at $x_1 \to x_2$ as we are interested for translation between pt 1 & 2. by cond^n (a), $\frac{\partial x}{\partial \alpha} = 0$ so we have

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\,dx ,$$

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$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x \partial \alpha}\,dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\,dx + \frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\,dx$$

(integrating by parts)

$\frac{\partial f}{\partial y'} = u$, $\frac{\partial y}{\partial \alpha} = v$

Sudv: uv - ∫vdu

by cond^n (5) $\left.\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} = 0$ i.e. at end points

$$\text{So } \frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}\,dx \quad \checkmark$$

To obtain the stationary value we multiply the above eqn by $d\alpha$ and find the derivative at $\alpha = 0$ so that

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} d\alpha = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\left(\frac{\partial y}{\partial \alpha}\Big|_{\alpha=0}\right) d\alpha\,dx$$

Now $\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} d\alpha = \delta I$ and $\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0} d\alpha = \delta y$ represent the

(3) δ variation of I and y. With the above eqn reduce to $\delta I = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx = 0$

By eqn II $\delta I = 0$:

So $\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx = 0$.

As δy is arbitrary its coeff. must vanish for the above relation to be satisfied. i.e.

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \quad \text{— Euler-Lagrange Differential Equation} \quad * (5)$$

This is the essential condn for the path I to be and extremum.

* Eqn (5) is later found to be similar to Lagrange Equation of motion.

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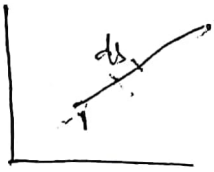
Application: 1) Geodesics – shortest distance between two points

Between two pts 1 & 2, the shortest distance is straight line and hence is an extremum. The eqn can be obtained by applying calculus of variation.



The arc length in a plane is given by $ds = (dx^2 + dy^2)^{1/2}$

or $ds = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = dx(1+\dot{y}^2)^{1/2}$ | $\frac{dy}{dx} = \dot{y}$

So the length of the curve between two points 1 & 2 is

$$I = \int_1^2 ds = \int_1^2 (1+\dot{y}^2)^{1/2} dx = \int_1^2 f\, dx \; ; \text{ where } f = (1+\dot{y}^2)^{1/2}$$

If the distance is shortest, the δ variation of the path has to vanish i.e $\delta I = 0$, which yields Lin eqn

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0} \quad \text{---(a)}$$

(4) With $f = (1+\dot{y}^2)^{1/2}$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}(1+\dot{y}^2)^{1/2} = \frac{1}{2}(1+\dot{y}^2)^{-1/2} \cdot 2\dot{y} = \frac{\dot{y}}{(1+\dot{y}^2)^{1/2}}$

Substituting the values in (a)

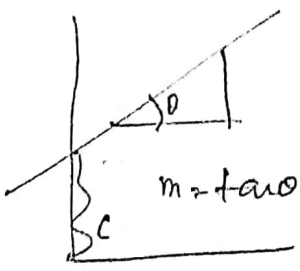
$$\cancel{\frac{\partial f}{\partial y}}^{0} - \frac{d}{dx}\left[\frac{\dot{y}}{(1+\dot{y}^2)^{1/2}}\right] = 0$$

or $\frac{d}{dx}\left[\frac{\dot{y}}{(1+\dot{y}^2)^{1/2}}\right] = 0$

or $\frac{\dot{y}}{(1+\dot{y}^2)^{1/2}} = \text{Const}$ ✓

This eqn is true only when

$\dot{y} = \text{const} = (a)$ say



$\Rightarrow \frac{dy}{dx} = m$ or $dy = m\,dx$ or

$\int dy = \int m\,dx + k$ (Const of integration)

or $y = mx + c$, represents a st line in a plane.

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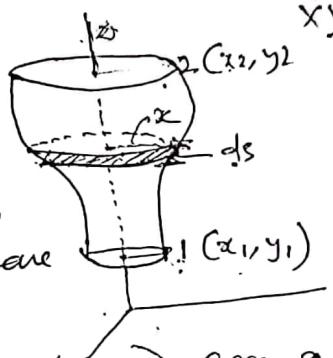
Minimum Surface of revolution.



Consider a curve extending from pt 1 to 2. If this curve is revolved about a certain axis, it forms a surface of revolution. Our aim is to find the minimum surface area for such revolution.

Consider a strip as shown by shaded area. It is formed by revolving the arc of length ds situated at a distance x from y-axis.

dA = Area of strip $= 2\pi x\, ds = 2\pi x (1+y'^2)^{1/2} dx$

So Total surface area $= \int dA = I = \int_1^2 2\pi x (1+y'^2)^{1/2} dx$ — (A)

will become minimum if $\delta I = 0$ or $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ — (B)

from (A), $f = x(1+y'^2)^{1/2}$

$\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial y'} = \frac{xy'}{(1+y'^2)^{1/2}}$

Substituting the values in (B)

$0 - \frac{d}{dx}\left(\frac{xy'}{(1+y'^2)^{1/2}}\right) = 0.$

$\Rightarrow \frac{xy'}{(1+y'^2)^{1/2}} = a$, a, Sing a const

$\Rightarrow \frac{x^2 y'^2}{1+y'^2} = a^2$ or $x^2y'^2 = a^2 + a^2y'^2$ or $y'^2 = \frac{a^2}{x^2-a^2}$

$\Rightarrow y' = \frac{a}{\sqrt{x^2-a^2}}$ or $\frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$ or $y = a\int \frac{dx}{\sqrt{x^2-a^2}} + b$

or $y = a\cosh^{-1}\frac{x}{a} + b$ or $\frac{y-b}{a} = \cosh^{-1} x/a$.

or $\frac{x}{a} = \cosh\left(\frac{y-b}{a}\right)$ or $\boxed{x = a\cosh\left(\frac{y-b}{a}\right)}$

This eqn is the eqn of a catenary.

∩ i.e. the Catenary

Hamilton's Variational Principle:

System point & Configuration space:

For motion of a single particle in space we need 3 co-ordinates and hence its space is called 3 dimensional space. For motion of two particle we need 6 co-ordinates 3 for each. But this system can be reduced to a single particle system as if the reduced particle is moving in a 6 dimensional space. Similarly a system of N particle may be considered to a single particle moving in a 3N dimensional space. This reduced single particle is called System point and this 3N dimensional space → as Configuration space. (Configuration space are not real)

Principle: Consider the motion of a mechanical system whose T → kinetic energy, V → potential energy. (f^n of co-ordinates & its derivatives) (f^n of co-ordinates alone and hence a conservative system)

for such cases the integral

$$\int_1^2 (T-V)\,dt$$ shall have a stationary value

we define a new parameter $L = T - V = T(q_j, \dot{q}_j) - V(q$

Statement of Hamilton Principle:

In a conservative system, the motion of the system from time t_1 to t_2 is such that for the integral

$$I = \int_{t_1}^{t_2} L\,dt \;, \quad L = T - V$$

is extremum for the path of motion.

Derivation of Lagranges Equation from Hamilton's Principle:

The parameter $L = T - V$ is called Lagrangian and is a f^n of, co-ordinate q_j, velocity $\dot{q}_j$ and time t.

i.e. $$L = L(q_j, \dot{q}_j, t) \qquad \dot{q}_j = \frac{dq_j}{dt}$$

According to Hamiltons Principle, the motion of a conserved system between time t_1 and t_2 takes place in such a way that the integral

$$I = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt$$ is an extremum.

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The δ-variation of this integral is zero i.e.

$$\delta I = 0$$

$$\Rightarrow \quad \delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0.$$

Considering the motion in the configuration space, each possible path is labelled by parameter α. With this the co-ordinates q_j and $\dot{q}_j$ are both fn of α and time 't' i.e.

$$q_j \to q_j(\alpha, t), \quad \dot{q}_j = \dot{q}_j(\alpha, t).$$

This makes the integral I dependent on α i.e. $I \to I(\alpha)$.

So $I(\alpha) = \int_{t_1}^{t_2} L(q(\alpha), \dot{q}(\alpha), t)\, dt$

Differentiating with respect to α

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha} + \frac{\partial L}{\partial t}\cancel{\frac{\partial t}{\partial \alpha}}^{\,0} \right) dt$$

(7) (In δ variation there is no time variation along any path i.e. $\frac{\partial t}{\partial \alpha} = 0$) So we have

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial^2 q_j}{\partial t \partial \alpha} d\alpha \right.$$

$$\therefore \frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \sum_j \frac{\partial L}{\partial \dot{q}_j}\cancel{\frac{\partial q_j}{\partial \alpha}}^{\,0} d\alpha \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{\partial q_j}{\partial \alpha} dt$$

(integrating by parts)

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] \frac{\partial q_j}{\partial \alpha} d\alpha dt \quad \Big| \quad \text{the value of } \frac{\partial q_j}{\partial \alpha}\Big|_{t_1}^{t_2} = 0 \text{ at } t_1 \& t_2.$$

$$\therefore \delta I(\alpha) = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right) \delta q_j\, dt.$$

As q_j are independent of each other the δ-variation δq_j will also be independent. This leads to $\delta I(\alpha) = 0$

i.e. $$\int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] \delta q_j\, dt = 0.$$

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✓ This is done only when the co-effs of δq_j s vanish i.e.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

or $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad j = 1, 2, \ldots 3N.$$

This equation is called Lagrange's equation of motion for a Conservative System.

+ D'Alembert's Principle:

This principle is based on differential method on principle of virtual work.

A system is subjected to a small displacement at any instant of time t in such a way that the configuration are independent of time. In other words there is no actual displacement and is termed as virtual displacement.

(8)

— Now let us consider a system of N particle in equilibrium. This suggest that the force on each particle $\vec{F}_i = 0$. If the particle undergo a small virtual displacement $\delta\vec{r}_i$, then the work done by this force is

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = 0 \quad \text{—— (1)}$$

$\delta\vec{r}_i$

d However the force $\vec{F}_i$ consist of two parts; the applied force $\vec{F}_i^a$ and the force of constraints i.e.

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i \quad \text{—— (1a)}$$

with this the above equation becomes

$$\sum_i \vec{F}_i^a \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0 \quad \text{—— (2)}$$

* For systems in which the virtual work done by force of constraints is zero i.e. $\vec{f}_i \cdot \delta\vec{r}_i = 0$.

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Then we have $\sum_i \vec{F}_i^a \cdot \delta\vec{r}_i = 0$. —(3)

An example → A particle is constrained to move on surface of a sphere that is highly smooth. For such cases the force of constraint is perpendicular to the surface. The virtual displacement is tangential to the surface. Angle between the two being 90°, the work done is zero).

Principle: Eqⁿ 3 is called principle of virtual work.

The state of equilibrium of a system was explained by D Alembert in a different way. He introduced the concept reversed force. He stated that a system can remain in equilibrium under action of a force which is sum of actual force $\vec{F}_i$ and a reversed effective force $-\dot{\vec{p}}_i$, so that

$$\vec{F}_i + (-\dot{\vec{p}}_i) = 0$$

or $$\vec{F}_i - \dot{\vec{p}}_i = 0 \quad \text{—(4)}$$

Here $-\dot{\vec{p}}_i$ is called reversed force of inertia on 'i'th particle. Further this equation tells that the external force $\vec{F}_i$ is supplemented by a reverse force $-\dot{\vec{p}}_i$. Using this the principle of virtual work given by eqⁿ (1) becomes

$$\boxed{\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0.}$$

using 1(a) $\sum_i \{(\vec{F}_i^a + \vec{f}_i) - \dot{\vec{p}}_i\} \cdot \delta\vec{r}_i = 0$

or $\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \delta\vec{r}_i = 0$

If the force of constraints are no more in picture

So the above eqⁿ becomes

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

or $$\boxed{\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0.} \quad \text{—(5)}$$

Superscript 'a' is now meaning less in absence of force of constraints.

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Eq"(5) is called D' Alemberts principle.

It may be noted here that $\delta\vec{r}_i$ are not independent of each other. So their coeff. cannot be equated to zero. It is therefore necessary to transform $\delta\vec{r}_i$'s to δq_j (generalised co-ordinates, which are ind. of each other. This will pave the way to equate co.eff. of each δq_j to zero.

Derivation of Lagrange's Equation from D'Alembert principle

The transformation equation from cartesian to generalised co-ordinate & vice versa may be given by

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \cdots q_n, t) \quad \text{---(6)}$$

Differentiating wrt to time

$$\frac{d\vec{r}_i}{dt} = \frac{\partial\vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial\vec{r}_i}{\partial t}\frac{dt}{dt}$$

(10)

$$\text{or } \vec{v}_i = \sum_{j=1} \frac{\partial\vec{r}_i}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial r_i}{\partial t} = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r}_i}{\partial t} \quad \text{---(7)}$$

An infinitesimal virtual displacement $\delta\vec{r}_i$ is given by

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j + \frac{\partial\vec{r}_i}{\partial t}\delta t$$

$$= \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \quad \text{---(8)}$$

The 2nd term is zero as virtual displacements are time independent

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

We have D' Alemberts equation

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0$$

using (8) for $\delta\vec{r}_i$

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i)\cdot\sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = 0$$

$$\text{or } \sum_{ij}\left(\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\right)\delta q_j - \sum_{ij}\dot{\vec{p}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = 0 \quad \text{---(9)}$$

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Using the notation for generalised force we have $\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = Q_j$, the above eqn becomes

$$\sum_j Q_j \cdot \delta q_j - \sum_{ij} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0 \quad \text{---(10)}$$

The second term on LHS of eqn (10) can be simplified as mentioned below

$$\sum_{ij} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \left(m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j$$

order of diff. interchanged

$$= \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \frac{\partial}{\partial q_j} \left(\frac{d\vec{r}_i}{dt} \right) \right] \delta q_j$$

$$= \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j$$

(11)

$$\therefore = \sum_{ij} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j$$

*Differentiation (+) wrt $\dot{q}_j$ we get $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m_i v_i^2 \right) \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \quad \text{---(11)}$$

$\left[\sum \frac{1}{2} m_i v_i^2 = T = KE. \right]$

using (11) in (10)

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

$$\Rightarrow \sum_j \left[Q_j - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0 \quad \text{---(12)}$$

Since q_j are independent of one another the above eqn (12) is true only when co-eff. of each δq_j vanish separately. This results in the eqn

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{---(13)}$$

This equation holds true for both conservative and non conservative system under the following consideration.

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Conservative System:

In such System the forces $\vec{F}_i$ are derivable from potential function V, which itself a function of co-ordinates ($\because V = V(\vec{r})$).

hence $\vec{F}_i = -\vec{\nabla}_i V = -\frac{\partial V}{\partial \vec{r}_i}$.

$V(q, \dot{q})$ The generalised force $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

$$= -\sum_i \vec{\nabla}_i V_i \frac{\partial r_i}{\partial q_j}$$

$$= -\sum \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial q_j}$$

Using this in (13)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$T - V = L$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

(12) $$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$ || Since potential V is ind. of $\dot{q}_j$

Recognise $T - V = L$ as the Lagrangian for the Conservative System so we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad (14)$$

Eq^n (14) is the Lagrange's Equation for Conservative Syst.

Non conservative System:

In these Systems, potentials are velocity dependent i.e. $U(q_j, \dot{q}_j)$ and are called generalised potential. The generalised force Q_j are given in terms of these potential as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \quad (15)$$

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using (15) in (13) we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

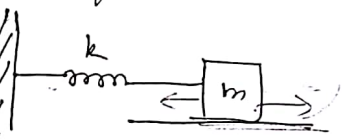
The lagrangian $L = T - U$. (for non conservative system)

The above equation becomes

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad \text{--- (16) Shri}$$

Lagranges Eq^n for non. conservative Systems.

Ex - Linear harmonic oscillator:



Sol^n 1-Dimensional System in x

Sol^n Lagranges eq^n becomes $F = -kx$

(13)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{---} \quad *$$

$V = \frac{dx}{dt} = \dot{x}$

We have KE $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$$PE\ V = -\int \vec{F}\cdot d\vec{x} = -\int -kx\,dx = \frac{1}{2}kx^2 + c$$

$\therefore$ Lagrangian $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

So $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial x} = -kx$.

Using this results in *

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

(1-1)

$\Rightarrow m\ddot{x} + kx = 0$ Shri required eq^n of SHO.

The frequency of oscillation is given by $\omega = \sqrt{k/m}$

$$2\pi f = \sqrt{k/m} \Rightarrow f = \frac{1}{2\pi}\sqrt{k/m}$$

$\Rightarrow \frac{1}{T} = \frac{1}{2\pi}\sqrt{k/m}$ $\Rightarrow T = 2\pi\sqrt{m/k}$

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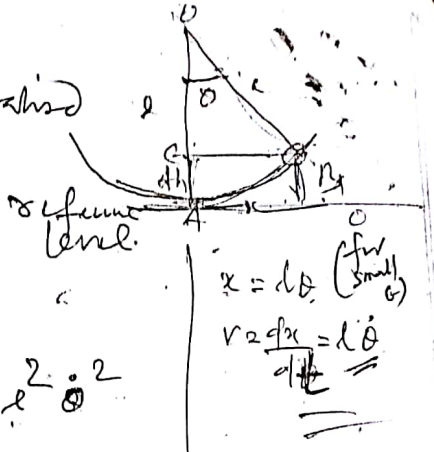
Simple pendulum

The angle of twist θ is considered generalised co-ordinate.



So the Lagranges eqn becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \longrightarrow (*)$$

Here $T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot\theta)^2 = \frac{1}{2}ml^2\dot\theta^2$

PE $V = mg(\Delta h) = mg(OA - OC) = mg(l - l\cos\theta)$

$$V = mgl(1-\cos\theta)$$

The Lagrangian $L = T - V = \frac{1}{2}ml^2\dot\theta^2 - mgl(1-\cos\theta)$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \quad \frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta$$

Using these results in (*) we get

(14) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$ becomes

$$\frac{d}{dt}(ml^2\dot\theta) - (-mgl\sin\theta) = 0$$

$$\Rightarrow ml^2\ddot\theta + mgl\sin\theta = 0$$

$$\Rightarrow \ddot\theta + g/l \sin\theta = 0$$

If θ is small, $\sin\theta \simeq \theta$ and the equation becomes

$$\ddot\theta + g/l\,\theta = 0$$ this eqn of SHM with $\omega^2 = g/l$

The time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{l/g}$

$\longrightarrow (*) \longrightarrow$

10 Sheets

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